

Global Mild Solutions of the Navier-Stokes Equations *

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Abstract

Here we establish a global well-posedness of *mild* solutions to the three-dimensional incompressible Navier-Stokes equations if the initial data are in the space \mathcal{X}^{-1} defined by (1.3) and if the norms of the initial data in \mathcal{X}^{-1} are bounded exactly by the viscosity coefficient μ .

Keyword: Navier-Stokes equations, global well-posedness, mild solutions.

1 The Result

The incompressible Navier-Stokes equations in $\mathbb{R}_+ \times \mathbb{R}^3$ are:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \mu \Delta v, & t > 0, x \in \mathbb{R}^3, \\ \nabla \cdot v = 0, & t > 0, x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where v is the velocity field of the fluid, p is the pressure and the constant μ is the viscosity. To solve the Navier-Stokes equations (1.1) in $\mathbb{R}_+ \times \mathbb{R}^3$, one assumes that the initial data

$$v(0, x) = v_0(x) \quad (1.2)$$

are divergent free and possess certain regularity.

The global existence of weak solutions goes back to Leray [9] and Hopf [6]. The global well-posedness of strong solutions for small initial data is due to Fujita and Kato [4] (see also Chemin [2]) in the Sobolev spaces \dot{H}^s , $s \geq \frac{1}{2}$, Kato [7] in the Lebesgue space $L^3(\mathbb{R}^3)$, and Koch and Tataru [8] in the space BMO^{-1} (see also [1] and [11]). It

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should be noted, in all these works, that the norms in corresponding spaces of the initial data are assumed to be very small, say they are smaller than the viscosity coefficient μ multiplied by a tiny positive constant ϵ .

In this note, we shall prove a new global well-posedness result for the three-dimensional incompressible Navier-Stokes equations. Our *mild* solutions will be in a scale-invariant function space which is natural with respect to the scalings of the Navier-Stokes equations. Moreover, we show that it is sufficient to assume the norms of the initial data to be less than the viscosity coefficient μ . The function space we will use here is

$$\mathcal{X}^{-1} = \{f \in \mathcal{D}'(R^3) : \int_{\mathbb{R}^3} |\xi|^{-1} |\widehat{f}| d\xi < \infty\}. \quad (1.3)$$

Here $\mathcal{D}'(R^3)$ represents the space of distributions and \widehat{f} represents the Fourier transform of f . The norm of \mathcal{X}^{-1} will be denoted by $\|\cdot\|_{\mathcal{X}^{-1}}$. We will also use the notation

$$\mathcal{X}^1 = \{f \in \mathcal{D}'(R^3) : \int_{\mathbb{R}^3} |\xi| |\widehat{f}| d\xi < \infty\}.$$

The norm of \mathcal{X}^1 is denoted by $\|\cdot\|_{\mathcal{X}^1}$.

Theorem 1.1. *The three-dimensional incompressible Navier-Stokes equation (1.1) is well-posed globally in time for the initial data in \mathcal{X}^{-1} satisfying*

$$\|v_0\|_{\mathcal{X}^{-1}} < \mu. \quad (1.4)$$

Moreover, the solution v is in $C(\mathbb{R}_+; \mathcal{X}^{-1}) \cap L^1(\mathbb{R}_+; \mathcal{X}^1)$ and satisfies

$$\sup_{0 \leq t < \infty} \left(\|v(t)\|_{\mathcal{X}^{-1}} + (\mu - \|v_0\|_{\mathcal{X}^{-1}}) \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right) \leq \|v_0\|_{\mathcal{X}^{-1}}.$$

Let us remark that the space BMO^{-1} is the largest space which is included in the tempered distribution and translation and scaling invariant, (see, for instance, the paper by Chemin and Gallagher [3]). Our space \mathcal{X}^{-1} is contained in BMO^{-1} . In fact, for each $f \in \mathcal{X}^{-1}$, write $f = \nabla \cdot (\nabla \Delta^{-1} f)$. It is clear that $\|\nabla \Delta^{-1} f\|_{BMO} \leq \|\nabla \Delta^{-1} f\|_{L^\infty} \leq \int |\xi|^{-1} |\widehat{f}| d\xi = \|f\|_{\mathcal{X}^{-1}}$. Hence, by Theorem 1 in [8], one has $f \in BMO^{-1}$. Let us also mention the classical example of the initial data given in [3]

$$u_0^\epsilon(x) = \frac{1}{\epsilon} \cos \frac{x_3}{\epsilon} (\partial_2 \phi, -\partial_1 \phi, 0)^T.$$

Due to Lemma 3.1 in [3], it is easy to see that $\|u_0^\epsilon\|_{\mathcal{X}^{-1}}$ is uniformly bounded independent of ϵ .

It is not hard to see that if the initial data are in the Sobolev space H^s , $s > \frac{1}{2}$, then they are also in the space \mathcal{X}^{-1} . However, this turns out to be false for $s = \frac{1}{2}$. An easy counterexample is given as follows: Let f be a non-negative function in the Schwartz

class $\mathcal{S}(\mathbb{R}^3)$ which is supported in the set $\{\xi \in \mathbb{R}^3 : 1 < |\xi| < 2\}$. Consider g which is defined by

$$\widehat{g} = \sum_{j \geq 1} \frac{2^{-2j}}{j} f(2^{-j}\xi).$$

It is clear that $g \in \dot{H}^{\frac{1}{2}}$ but

$$\|g\|_{\mathcal{X}^{-1}} = \sum_{j \geq 1} \int \frac{2^{-3j}}{j} f(2^{-j}\xi) d\xi = \|f\|_{L^1} \sum_{j \geq 1} \frac{1}{j} = \infty.$$

2 Proof of Theorem

Let ζ be the standard mollifier in \mathbb{R}^3 : $\zeta \in C_0^\infty$, $0 \leq \zeta \leq 1$, $\int \zeta(x) dx = 1$. For $\lambda > 0$, let $\zeta^\lambda(x) = \lambda^{-3} \zeta(\lambda^{-1}x)$ and $v_0^\lambda = \zeta^\lambda * v_0$. For $v_0 \in \mathcal{X}^{-1}$, since $|\widehat{\zeta}(\xi)| \leq \int \zeta(x) dx = 1$, one has

$$\begin{cases} \|v_0^\lambda\|_{\mathcal{X}^{-1}} = \int |\xi|^{-1} |\widehat{v_0}(\xi)| |\widehat{\zeta}(\lambda\xi)| d\xi \leq \|v_0\|_{\mathcal{X}^{-1}}, \\ \|v_0^\lambda\|_{L^\infty} \lesssim \int |\xi| |\widehat{\zeta}(\lambda\xi)| |\xi|^{-1} |\widehat{v_0}(\xi)| d\xi \leq C_\lambda \|v_0\|_{\mathcal{X}^{-1}}. \end{cases} \quad (2.1)$$

Consequently, by the standard local existence theory of the Navier-Stokes equations, there exists a unique local smooth solution $v^\lambda(t, x)$ on some time interval $[0, T_\lambda)$. The associated pressure p^λ is given by $p^\lambda = (R \otimes R) : (v \otimes v)$ with R being the Riesz operator (see, for instance, [5]).

We need to derive an *a priori* estimate under the condition (1.4). First of all, it is easy to see that

$$\|v^\lambda\|_{\mathcal{X}^{-1}} = \int_{|\xi| \leq 1} |\xi|^{-1} |\widehat{v^\lambda}| d\xi + \int_{|\xi| > 1} |\xi|^{-2} |\widehat{\sqrt{-\Delta} v^\lambda}| d\xi \leq C \|v^\lambda\|_{H^1} \quad (2.2)$$

and

$$\begin{aligned} \int_0^{T_\lambda} \|v^\lambda(t)\|_{\mathcal{X}^1} dt &= \int_0^{T_\lambda} \int_{|\xi| \leq 1} |\xi| |\widehat{v^\lambda}| d\xi dt \\ &+ \int_0^{T_\lambda} \int_{|\xi| > 1} |\xi|^{-2} |\widehat{\sqrt{-\Delta} v^\lambda}| d\xi dt \leq C \int_0^{T_\lambda} \|v^\lambda\|_{H^3} ds. \end{aligned} \quad (2.3)$$

Hence, one has $v^\lambda \in L^\infty(0, T_\lambda; \mathcal{X}^{-1}) \cap L^1(0, T_\lambda; \mathcal{X}^1)$ (without uniform norms bound in λ). Next, let us take the Fourier transform of (1.1) to get

$$\begin{cases} \partial_t \widehat{v^\lambda} - i \int \widehat{v^\lambda}(\eta) \otimes \widehat{v^\lambda}(\xi - \eta) d\eta \cdot \xi - i \xi p^\lambda + \mu |\xi|^2 \widehat{v^\lambda}, \\ \xi \cdot \widehat{v^\lambda} = 0. \end{cases} \quad (2.4)$$

From (2.4), (2.2) and (2.3), we deduce that

$$\begin{aligned}
& \partial_t \int |\xi|^{-1} |\widehat{v^\lambda}| d\xi + \mu \int |\xi| |\widehat{v^\lambda}| d\xi \\
&= \frac{i}{2} \iint \left([\widehat{v^\lambda}(\eta) \cdot |\widehat{v^\lambda}(\xi)|^{-1} \overline{\widehat{v^\lambda}(\xi)}] \widehat{v^\lambda}(\xi - \eta) \right. \\
&\quad \left. - [\overline{\widehat{v^\lambda}(\eta)} \cdot |\widehat{v^\lambda}(\xi)|^{-1} \widehat{v^\lambda}(\xi)] \overline{\widehat{v^\lambda}(\xi - \eta)} \right) \cdot |\xi|^{-1} \xi d\eta d\xi \\
&\leq \int \int |\widehat{v^\lambda}(\eta)| |\widehat{v^\lambda}(\xi - \eta)| d\eta d\xi \\
&\leq \frac{1}{2} \int \int (|\eta|^{-1} |\xi - \eta| + |\eta| |\xi - \eta|^{-1}) |\widehat{v^\lambda}(\eta)| |\widehat{v^\lambda}(\xi - \eta)| d\eta d\xi \\
&\leq \int |\xi|^{-1} |\widehat{v^\lambda}(\xi)| d\xi \int |\xi| |\widehat{v^\lambda}(\xi)| d\xi.
\end{aligned} \tag{2.5}$$

By (1.4) and (2.2), we see that $\|v^\lambda(t)\|_{\mathcal{X}^{-1}} < \mu$ at least for a very short time interval $[0, \delta]$ with $0 < \delta < T_\lambda$. Consequently, on such a time interval, one has

$$\partial_t \|v^\lambda\|_{\mathcal{X}^{-1}} \leq 0, \quad \text{hence } \|v^\lambda\|_{\mathcal{X}^{-1}} \leq \|v_0\|_{\mathcal{X}^{-1}} < \mu.$$

A continuity argument in the time variable yields that

$$\|v^\lambda(t)\|_{\mathcal{X}^{-1}} \leq \|v_0\|_{\mathcal{X}^{-1}} < \mu$$

for all $t \in [0, T_\lambda)$. We then apply (2.5) once more to derive that

$$\|v^\lambda(t)\|_{\mathcal{X}^{-1}} + (\mu - \|v_0\|_{\mathcal{X}^{-1}}) \int_0^t \|v^\lambda(s)\|_{\mathcal{X}^1} ds \leq \|v_0\|_{\mathcal{X}^{-1}}, \quad \text{for all } t \in [0, T_\lambda). \tag{2.6}$$

As a bi-product of (2.6), one obtains that

$$\int_0^{T_\lambda} \|\nabla v^\lambda(t)\|_{L^\infty} dt \leq \int_0^{T_\lambda} \|v^\lambda(t)\|_{\mathcal{X}^1} dt \leq \frac{\|v_0\|_{\mathcal{X}^{-1}}}{\mu - \|v_0\|_{\mathcal{X}^{-1}}}.$$

The standard energy method ([10]) gives that

$$\|v^\lambda(t)\|_{H^k} \leq \|v_0\|_{H^k} \exp \left\{ c_k \int_0^{T_\lambda} \|\nabla v^\lambda(s)\|_{L^\infty} ds \right\} \leq \|v_0\|_{H^k} \exp \left\{ \frac{c_k \|v_0\|_{\mathcal{X}^{-1}}}{\mu - \|v_0\|_{\mathcal{X}^{-1}}} \right\}$$

for all $0 \leq t < T_\lambda$ and all $k > 0$. The latter implies that $T_\lambda = \infty$. Moreover, one has the uniform estimate for v^λ :

$$\sup_{0 \leq t < \infty} \left(\|v^\lambda(t)\|_{\mathcal{X}^{-1}} + (\mu - \|v_0\|_{\mathcal{X}^{-1}}) \int_0^t \|v^\lambda(s)\|_{\mathcal{X}^1} ds \right) \leq \|v_0\|_{\mathcal{X}^{-1}}, \tag{2.7}$$

under the condition (1.4).

The estimate (2.7) implies that there exists a subsequence of $\{v^\lambda\}$ (we will still denote it by $\{v^\lambda\}$) such that, as $\lambda \rightarrow 0$,

$$v^\lambda \rightharpoonup v \quad \text{in } L^1(\mathbb{R}_+; \mathcal{X}^1), \quad v^\lambda \rightharpoonup v \quad \text{weakly}^* \text{ in } L^\infty(\mathbb{R}_+; \mathcal{X}^{-1}) \quad (2.8)$$

for some

$$v \in L^\infty(\mathbb{R}_+; \mathcal{X}^{-1}) \cap L^1(\mathbb{R}_+; \mathcal{X}^1). \quad (2.9)$$

For the initial data, we note that

$$\begin{aligned} \|v_0^\lambda - v_0\|_{\mathcal{X}^{-1}} &= \int_{|\xi| \leq M} |\xi|^{-1} |\widehat{\zeta}(\lambda\xi) - 1| |\widehat{v}_0(\xi)| d\xi \\ &\quad + \int_{|\xi| > M} |\xi|^{-1} |\widehat{\zeta}(\lambda\xi) - 1| |\widehat{v}_0(\xi)| d\xi \\ &\leq 2 \sup_{|\eta| \leq \lambda M} |\widehat{\zeta}(\eta) - 1| \int_{|\xi| < M} |\xi|^{-1} |\widehat{v}_0(\xi)| d\xi + 2 \int_{|\xi| > M} |\xi|^{-1} |\widehat{v}_0(\xi)| d\xi. \end{aligned}$$

By taking $M = \lambda^{-\frac{1}{2}}$, and using the identity $\widehat{\zeta}(0) = \int \zeta(x) dx = 1$, one concludes that

$$\|v_0^\lambda - v_0\|_{\mathcal{X}^{-1}} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0. \quad (2.10)$$

To show the strong convergence of v^λ , we proceed similarly as in (2.5) to calculate that

$$\begin{aligned} &\partial_t \int |\xi|^{-1} |\widehat{v}^{\lambda_1} - \widehat{v}^{\lambda_2}| d\xi + \mu \int |\xi| |\widehat{v}^{\lambda_1} - \widehat{v}^{\lambda_2}| d\xi \\ &\leq \iint (|\widehat{v}^{\lambda_1}(\eta)| + |\widehat{v}^{\lambda_2}(\eta)|) |\widehat{v}^{\lambda_1}(\xi - \eta) - \widehat{v}^{\lambda_2}(\xi - \eta)| d\eta d\xi \\ &\leq \frac{1}{2} \iint (|\eta|^{-1} |\xi - \eta| + |\eta| |\xi - \eta|^{-1}) \\ &\quad \times (|\widehat{v}^{\lambda_1}(\eta)| + |\widehat{v}^{\lambda_2}(\eta)|) |\widehat{v}^{\lambda_1}(\xi - \eta) - \widehat{v}^{\lambda_2}(\xi - \eta)| d\eta d\xi \\ &\leq \frac{1}{2} (\|v^{\lambda_1}\|_{\mathcal{X}^{-1}} + \|v^{\lambda_2}\|_{\mathcal{X}^{-1}}) \|v^{\lambda_1} - v^{\lambda_2}\|_{\mathcal{X}^1} \\ &\quad + \frac{1}{2} (\|v^{\lambda_1}\|_{\mathcal{X}^1} + \|v^{\lambda_2}\|_{\mathcal{X}^1}) \|v^{\lambda_1} - v^{\lambda_2}\|_{\mathcal{X}^{-1}}. \end{aligned}$$

Combining the above with (2.7), we obtain that

$$\begin{aligned} &\partial_t \int |\xi|^{-1} |\widehat{v}^{\lambda_1} - \widehat{v}^{\lambda_2}| d\xi + (\mu - \|v_0\|_{\mathcal{X}^{-1}}) \int |\xi| |\widehat{v}^{\lambda_1} - \widehat{v}^{\lambda_2}| d\xi \\ &\leq \frac{1}{2} (\|v^{\lambda_1}\|_{\mathcal{X}^1} + \|v^{\lambda_2}\|_{\mathcal{X}^1}) \|v^{\lambda_1} - v^{\lambda_2}\|_{\mathcal{X}^{-1}}. \end{aligned}$$

The latter implies further that

$$\begin{cases} \|v^{\lambda_1}(t) - v^{\lambda_2}(t)\|_{\mathcal{X}^{-1}} \leq \|v_0^{\lambda_1} - v_0^{\lambda_2}\|_{\mathcal{X}^{-1}} \exp \left\{ \frac{\|v_0\|_{\mathcal{X}^{-1}}}{\mu - \|v_0\|_{\mathcal{X}^{-1}}} t \right\}, \\ (\mu - \|v_0\|_{\mathcal{X}^{-1}}) \int_0^\infty \|v^{\lambda_1} - v^{\lambda_2}\|_{\mathcal{X}^1} dt \leq \|v_0^{\lambda_1} - v_0^{\lambda_2}\|_{\mathcal{X}^{-1}} \exp \left\{ \frac{\|v_0\|_{\mathcal{X}^{-1}}}{\mu - \|v_0\|_{\mathcal{X}^{-1}}} \right\}. \end{cases} \quad (2.11)$$

Combining (2.10) and (2.11), we conclude that $\{v^\lambda\}$ is a Cauchy sequence in $L^\infty(\mathbb{R}_+; \mathcal{X}^{-1}) \cap L^1(\mathbb{R}_+; \mathcal{X}^1)$ and the convergence in (2.8) is a strong one. In fact, (2.11) also yields the uniqueness of solutions in the space $L^\infty(\mathbb{R}_+; \mathcal{X}^{-1}) \cap L^1(\mathbb{R}_+; \mathcal{X}^1)$ under the assumption (1.4).

To get the further time regularity of $v(t, x)$, we come back to the equation (2.4). We claim that $\partial_t v^\lambda$'s are uniformly bounded in $L^1(\mathbb{R}_+; \mathcal{X}^{-1})$. Indeed, by (2.7), it is obvious that Δv^λ are uniformly bounded in $L^1(\mathbb{R}_+; \mathcal{X}^{-1})$. Moreover, as calculated in (2.5), one has

$$\begin{aligned} \|\nabla \cdot (v^\lambda \otimes v^\lambda)\|_{L^1(\mathcal{X}^{-1})} &\leq \int_0^\infty \iint |\widehat{v}^\lambda(\eta)| |\widehat{v}^\lambda(\xi - \eta)| d\xi d\eta dt \\ &\leq \int_0^t \|v^\lambda\|_{\mathcal{X}^{-1}} \|v^\lambda\|_{\mathcal{X}^1} dt \leq \sup_t \|v^\lambda(t)\|_{\mathcal{X}^{-1}} \|v^\lambda\|_{L^1(\mathcal{X}^1)}. \end{aligned}$$

The pressures can be treated by the same way. We hence proved the claim

$$\partial_t v^\lambda \in L^1(\mathbb{R}_+; \mathcal{X}^{-1}), \quad \|v^\lambda\|_{L^1(\mathcal{X}^1)} \leq C_0 \|v_0\|_{\mathcal{X}^{-1}} (1 + \|v_0\|_{\mathcal{X}^{-1}}). \quad (2.12)$$

The latter allows us to improve (2.9) and to finally conclude

$$v \in C(\mathbb{R}_+; \mathcal{X}^{-1}) \cap L^1(\mathbb{R}_+; \mathcal{X}^1), \quad \partial_t v \in L^1(\mathbb{R}_+; \mathcal{X}^{-1}). \quad (2.13)$$

Remark 2.1. Let us remark that a similar estimate as (2.5) also implies a type of Beale-Kato-Majda's criterion of Navier-Stokes equations: if

$$\int_0^T \int |\widehat{\omega}(\xi)| d\xi dt < \infty,$$

then a smooth solution on $[0, T)$ can be extended to $[0, T + \delta)$ for some $\delta > 0$. In fact,

$$\partial_t \int |\xi|^s |\widehat{v}| d\xi + \mu \int |\xi|^{s+2} |\widehat{v}| d\xi \leq \iint |\xi|^{s+1} |\widehat{v}(\eta)| |\widehat{v}(\xi - \eta)| d\eta d\xi.$$

Noting $v = \Delta^{-1} \text{curl} \omega$ and taking $s = -1$ and 0, one gets that

$$\|v(t)\|_{\mathcal{X}^{-1}} \leq \|v_0\|_{\mathcal{X}^{-1}} \exp \left\{ \int_0^t \int |\widehat{\omega}(\xi)| d\xi ds \right\} < \infty$$

and

$$\|v(t)\|_{\mathcal{X}^0} \leq \|v_0\|_{\mathcal{X}^0} \exp \left\{ 2 \int_0^t \int |\widehat{\omega}(\xi)| d\xi ds \right\} < \infty$$

for all $0 \leq t < T$. Then for $s > 0$ one has

$$\begin{aligned} \partial_t \int |\xi|^s |\widehat{v}| d\xi + \mu \int |\xi|^{s+2} |\widehat{v}| d\xi &\leq \iint_{|\xi| \leq M_s} |\xi|^{s+1} |\widehat{v}(\eta)| |\widehat{v}(\xi - \eta)| d\eta d\xi \\ &\quad + \epsilon_s \iint_{|\xi| > M_s} (1 + |\eta|^{s+2} + |\xi - \eta|^{s+2}) |\widehat{v}(\eta)| |\widehat{v}(\xi - \eta)| d\eta d\xi \\ &\leq (M_s^{s+1} + \epsilon_s) \|v\|_{\mathcal{X}^{-1}} \|v\|_{\mathcal{X}^1} + 2\epsilon_s \int |\widehat{v}| d\xi \int |\xi|^{s+2} |\widehat{v}(\xi)| d\xi. \end{aligned}$$

Here $\epsilon_s > 0$ is a small constant such that $2\epsilon_s \int |\widehat{v}_0| d\xi \exp\{2 \int_0^T \int |\widehat{\omega}(\xi)| d\xi dt\} < \mu$ which implies $2\epsilon_s \int |\widehat{v}| d\xi < \mu$. Then $M_s > 1$ is chosen to be a large constant such that $|\xi|^{s+1} \leq \epsilon_s(1+|\eta|^{s+2}+|\xi-\eta|^{s+2})$ for $|\xi| > M_s$. Consequently, one has $\|v(t)\|_{C^s} < \infty$. We emphasize that the bounds of $\|v\|_{C^s}$ are only one exponential in terms of $\int_0^T \int |\widehat{\omega}(\xi)| d\xi dt$.

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References

- [1] M. Cannone, *Ondelettes, paraproducts et Navier-Stokes*. Diderot editeur, Arts et Sciences, 1995.
- [2] J. M. Chemin, *Remarques sur l'existence globale pour le système de Navier-Stokes incompressible*. SIAM Journal on Mathematical Analysis 1992; 23:20–28.
- [3] J. M. Chemin and I. Gallagher, *Wellposedness and stability results for the Navier-Stokes equations in \mathbb{R}^3* . Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 2, 599–624
- [4] H. Fujita and T. Kato *On the Navier-Stokes initial value problem I*. Archive for Rational Mechanics and Analysis 1964; 16:269–315.
- [5] Y. Giga, K. Inui and S. Matsui, *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data*. Advances in fluid dynamics, 27C68, Quad. Mat., 4, Dept. Math., Seconda Univ. Napoli, Caserta, 1999.
- [6] Hopf, E., *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachrichten 4(1950-51), 213–231.
- [7] T. Kato, *Strong L^p solutions of the Navier-Stokes equations in \mathbb{R}^m , with applications to weak solutions*. Mathematische Zeitschrift 1984; 187:471–480.
- [8] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*. Advance Mathematics 2001; 157:22–35.
- [9] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. 63 (1934), no. 1, 193–248.

- [10] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*. Cambridge Texts in Applied Mathematics, 27. Cambridge University, Cambridge, Mass., 2002.
- [11] F. Planchon, *Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in \mathbb{R}^3* . Annales l'Institut Henri Poincare 1996; 13:319–336.